CONVERGENCE RATES AND DECOUPLING IN LINEAR STOCHASTIC APPROXIMATION ALGORITHMS

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Abstract. Almost sure convergence rates for linear algorithms $h_{k+1} = h_k + \frac{1}{k^\chi}(b_k - A_k h_k)$ are studied, where $\chi \in (0,1)$, $\{A_k\}_{k=1}^\infty$ are symmetric, positive semidefinite random matrices and $\{b_k\}_{k=1}^\infty$ are random vectors. It is shown that $n^\gamma |h_n - A^{-1}b| \to 0$ a.s. for the $\gamma \in [0,\chi)$, positive definite A and vector b such that $\frac{1}{n^{\chi-\gamma}} \sum_{k=1}^{n} (A_k - A) \to 0$ and $\frac{1}{n^{\chi-\gamma}} \sum_{k=1}^{n} (b_k - b) \to 0$ a.s. When $\chi - \gamma \in \left(\frac{1}{2}, 1\right)$, these assumptions are implied by the Marcinkiewicz strong law of large numbers, which allows the $\{A_k\}$ and $\{b_k\}$ to have heavy-tails, long-range dependence or both. Finally, corroborating experimental outcomes and decreasing-gain design considerations are provided.

Key words. rates of convergence, stochastic approximation, Marcinkiewicz strong law of large numbers, heavy tails, long-range dependence, linear process.

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1. Introduction. Linear stochastic approximation algorithms have found widespread application in parameter estimation, adaptive machine learning, signal processing, econometrics and pattern recognition (see, e.g., [1], [3], [9], [26] and [32]). Consequently, their asymptotic rates of almost sure and r^{th}-mean convergence as well as invariance and large deviation principles are of utmost importance (see e.g., [6], [11], [17], [18], [21], [22], [24], [34] and [36]). For motivation, suppose $\{x_k, k = 1, 2, \cdots\}$ and $\{y_k, k = 2, 3, \cdots\}$ are second order $\mathbb{R}^d-$ and $\mathbb{R}-$valued stochastic processes, defined on some probability space $(\Omega, \mathcal{F}, P)$, that satisfy

$$(1) \qquad y_{k+1} = x_k^T h + \epsilon_k, \qquad \forall k = 1, 2, \ldots,$$

where h is an unknown d-dimensional parameter or weight vector of interest and ϵ_k is a noise sequence. One often wants to find the value of h that minimizes the mean-square error $h \to E|y_{k+1} - x_k^T h|^2$. This *best* h is given by $h = A^{-1}b$, where $A = E(x_k x_k^T)$ and $b = E(y_{k+1} x_k)$, assuming the expectations exist, wide-sense stationarity conditions and that A is positive definite. However, we often do not know the joint distribution of (x_k, y_{k+1}) nor have the necessary stationarity but instead estimate h using a linear algorithm of the form:

$$(2) \qquad h_{k+1} = h_k + \mu_k(b_k - A_k h_k),$$

where μ_k is the kth step size (often of the form $\mu_k = k^{-\chi}$ for some $\chi \in \left(\frac{1}{2}, 1\right]$) and

$$(3) \qquad A_k = \frac{1}{N} \sum_{l=\max\{k-N+1,1\}}^{k} x_l x_l^T, \text{ and } b_k = \frac{1}{N} \sum_{l=\max\{k-N+1,1\}}^{k} y_{l+1} x_l$$

for some $N \in \mathbb{N}$, are random sequences of symmetric, positive-semi-definite matrices and vectors respectively. Most often $N = 1$ so $A_k = x_k x_k^T$ and $b_k = y_{k+1} x_k$. More information on stochastic approximation can be found in e.g. [5], [8], [10], [13],

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[17], [25], [30] and [37], which provide examples and motivation for our work. However, our work is easily differentiated from these. Delyon [8], for example, focuses on non-linear stochastic approximation algorithms, treating linear examples the same as non-linear ones. (In Section 4.2.2 he uses linear algorithm approximation but with a constant deterministic matrix $A_k = A$ in our notation.) Delyon's work handles important applications. However, his A-stable and (A, B) Conditions are usually harder to verify than our Marcinkiewicz Strong Law of Large Numbers (MSLLN) conditions (given below) in the (unbounded, random A_k) linear case, he does not supply almost sure rates of convergence, his theorems are geared to martingale-increment-plus-decreasing-perturbation noise and he often assumes fourth order moments. We are motivated by (but not restricted to) the common setting where $X_k^T = (x_k^T, y_{k+1})$ is a (multivariate) linear process

$$(4) \qquad\qquad X_k = \sum_{l=-\infty}^{\infty} C_{k-l} \Xi_l.$$

Matrix sequence (C_l) can decay slowly enough (as $|l| \to \infty$) for long-range dependence (LRD) while $\{\Xi_l\}$ can have heavy tails (HT), so $E|b_k|^2 = \infty$ and/or $E|A_k|^2 = \infty$. Even in the lighter tail, short-range dependence case our two-sided linear process example $\{x_k\}$ is not a martingale. Moreover; long-range dependence and heavy tails; exhibited in many network [19], financial and paleoclimatic data sets for example; voids the usual mixing and moment conditions. We focus on one-step versus Polyak-Ruppert's two-step averaging algorithms but handle heavy tails and long range dependence, deriving a surprising decoupling. This means that the optimal convergence rate of (2) is affected by either the heavy tails or the long-range dependence, whichever is worse, but not both. This contrasts the rate for partial sums of long-range dependent, heavily-tailed random variables, which is degraded twice (see e.g. Theorem 6).

Step size μ_k has a direct effect on the convergence rate and algorithm effectiveness (see, e.g [12], [15] and references cited therein). Consider the extreme cases. In the homogeneous, deterministic setting, i.e. $A_k = A$ and $b_k = b$, (2) can solve the linear equation $Ah = b$ when matrix inversion of A is ill-conditioned. In this case, a constant gain $\mu_k = \epsilon$ is best: Since $b = Ah$, we have $h_{k+1} = h_k - \epsilon A(h_k - h)$, so $h_n - h = (I - \epsilon A)^{n-1}(h_1 - h)$ and $h_n \to h$ geometrically, provided ϵ is small enough that the eigenvalues of $I - \epsilon A$ are within the unit disc. Conversely, in the presence of persistent noise, decreasing step sizes are required for the convergence $h_n \to h$. Existing results show that the best possible almost-sure rate of convergence is $|h_n - h| = O\left(\sqrt{n^{-1} \log\log(n)}\right)$, implied by the law of the iterated logarithm, and that this rate is only attainable when $\mu_k = \frac{1}{k}$, second moments of A_k, b_k exist and there is no long-range dependence. (These claims follow from the almost-sure invariance principle in Kouritzin [22].)

Herein, we handle all gains, long range dependence and heavy tails, addressing the optimal rate of convergence by establishing results akin to the MSLLN, namely $n^\gamma |h_n - h| \to 0$ a.s. (i.e. $|h_n - h| = o(n^{-\gamma})$), for all $\gamma < \gamma_0(\chi) \doteq \chi - M$. M is called the *Marcinkiewicz threshold* in the sequel and is defined by

$$(5) \qquad M \doteq \inf\left\{ \frac{1}{m} : \lim_{n\to\infty} \frac{1}{n^{\frac{1}{m}}} \sum_{k=1}^{n} (A_k - A) = 0, \ \lim_{n\to\infty} \frac{1}{n^{\frac{1}{m}}} \sum_{k=1}^{n} (b_k - b) = 0 \ \text{ a.s.} \right\}.$$

Usually, we expect $M \in (\frac{1}{2}, 1]$, due to Strong Law of Large Numbers and Central Limit Theorem in the light-tail, short-range-dependence case but when there is LRD

and/or HT M generally cannot approach $\frac{1}{2}$. When $\{(x_k^T, y_{k+1})^T : k \in \mathbb{Z}\}$ is a linear process as in (4), it is shown in [20] that $M = \frac{1}{\alpha} \vee (2 - 2\sigma)$ with $\alpha \doteq \sup\{a \leq 2 : \sup_{t \geq 0} t^a P(|\Xi_1|^2 > t) < \infty\}$ and $\sigma \doteq \sup\{s \in (\frac{1}{2}, 1] : \sup_l |l|^s \|C_l\| < \infty\}$. Hence, $\gamma < \gamma_0(\chi) \doteq (\chi - \frac{1}{\alpha}) \wedge (\chi + 2\sigma - 2)$. Here, $\alpha \in (1, 2]$ is a heavy-tail parameter with $\alpha = 2$ indicating non-heavy tails and $\sigma \in (\frac{1}{2}, 1]$ is a long-range dependence parameter with $\sigma = 1$ indicating the minimal amount of long-range dependence.

In classical applications the best theoretical convergence rate is attained when $\chi = 1$ corresponding to $\gamma_0(\chi) = \frac{1}{2}$. However, this rate knowledge can lead to erroneous conclusions as the algorithm often performs better with $\mu_k = k^{-\chi}$ for some $\chi < 1$ or even constant gain (see [23]) than with $\mu_k = \frac{1}{k}$. How might one explain this apparent paradox? First of all, these simple rate-of-convergence results do not account for the possibility of exploding constants, i.e. if $h_k^{(\chi)}$ denotes the solution of the algorithm (2) with $\mu_k = k^{-\chi}$, then $|h_n(\chi) - h| = D^\chi n^{-\gamma(\chi)}$ for all $\gamma(\chi) < \gamma_0(\chi)$. However, this D^χ often increases rapidly as $\chi \nearrow 1$ so the *observed* convergence may be fastest for some $\chi < 1$. Secondly, a higher value of χ is worse for forgetting a poor initial guess h_0 of h since you move further and further from the geometric convergence mentioned above as $\chi \to 1$.

Our approach is to transfer the MSLLN from the partial sums of a linear algorithm's coefficients to its solution. In other words, we establish the almost sure rates of convergence $n^\gamma |h_n - h| \to 0$ a.s., for the algorithm

$$(6) \qquad h_{k+1} = h_k + \frac{1}{k^\chi}(b_k - A_k h_k) \ \ \forall \ k = 1, 2, 3, ...$$

with $\chi \in (0, 1)$, assuming only

$$(7) \qquad \lim_{n \to \infty} \frac{1}{n^\chi} \sum_{k=1}^n (A_k - A) = 0 \ \ \text{and} \ \ \lim_{n \to \infty} \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^n (b_k - A_k h) = 0 \ \ \text{a.s.}$$

for some $\gamma \in [0, \chi)$, which can be implied by e.g.

$$(8) \qquad \lim_{n \to \infty} \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^n (A_k - A) = 0 \ \ \text{and} \ \ \lim_{n \to \infty} \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^n (b_k - b) = 0 \ \ \text{a.s.,}$$

where $Ah = b$. When $\chi - \gamma \in (\frac{1}{2}, 1]$, these conditions can be verified by the MSLLN under a variety of conditions, which we study using the specific structure of A_k and b_k in Section 4.

In addition to rates of convergence, our results show that convergence ($h_k \to h$) in (6) takes place provided that $\chi \in (M, 1)$. All this suggests that more quickly decreasing gains like $\mu_k = \frac{1}{k^\chi}$ with χ near 1 should be used in very heavy-tailed or long-range dependent settings. Conversely, slowly deceasing gains like $\mu = \frac{1}{k^\chi}$ with smaller χ might work well in lighter-tailed, short-range-dependent situations. Our simulations in Section 5 show that the smallest normalized error, $\frac{|h_n - h|}{|h_1 - h|}$, usually occurs for $\chi \in (M, 1]$ and the most commonly used choice $\chi = 1$ is most appropriate in very heavy-tailed or long-range-dependent settings (where M is close to 1) or very long runs. In other words, a slower decreasing gain usually gets you close to the true parameters h more quickly unless the coefficients have a high probability of differing significantly from their means.

Let us consider what is new in terms of our theoretical results. The idea of inferring convergence and rates of convergence results for linear algorithms (2) from

like convergence and rates of convergence of its coefficients is not new. Indeed, it dates back at least to work done by one of the authors in 1994 and 1996 (see [21], [22] and [24]). The result [24] considered relatively general gain μ_k and achieved optimal rates of rth-mean convergence. It has been proved in [22] that the solution of the linear algorithm (2) satisfies an almost sure invariance principle with respect to a limiting Gaussian process when $\mu_k = \frac{1}{k}$ and each A_k is symmetric under the minimal condition that the coefficients satisfy such an a.s. invariance principle. One could then immediately transfer functional laws of the iterated logarithm from the limiting Gaussian process back to the solution of the linear algorithm. Again assuming the "usual" conditions of A_k symmetry and $\mu_k = \frac{1}{k}$, Kouritzin [21] showed that the solution of the linear algorithm converges almost surely given that the coefficients do. While this result does not state rates of convergence, our current work in going from Proposition 10 to Theorem 1 within shows that almost-sure rate of convergence sometimes follow from convergence results for linear algorithms as a simple corollary.

There were many results (see, e.g. [13], [14] and [16]) that preceded those mentioned above and gave convergence or rates of convergence for linear algorithms. However, these results assumed a specific dependency structure and, thereby, were not generally applicable. More recently, some authors, e.g. [6], [8] and [34], have followed the path of transferring convergence and rates of convergence from partial sums of (the coefficient) random variables to the solutions of linear equations. Specifically, Tadić [34] transferred almost-sure rates of convergence, including those of the law-of-the-iterated-logarithm rate, from the coefficients to the linear algorithm in the non-symmetric-A_k, general-gain case. He does not develop a law of the iterated logarithm where one characterizes the limit points nor does he consider functional versions. Moreover, he imposes one of two sets of conditions (A and B in his notation). Conditions B ensure the gain $\mu_k \approx \frac{1}{k}$, so these results should be compared to prior results in [4] and [22], which imply stronger Strassen-type functional laws of the iterated logarithm. Tadić does not give any examples verifying his Conditions A where lessor rates are obtained.

It seems that we are the first to consider processes that are simultaneously heavy-tailed and long-range dependent in stochastic approximation.

The rest of this paper is organized as follows. A motivational example is given next. The main theorems are formulated in Section 3. Then, Section 4 includes some background about the Marcinkiewicz Strong Law of Large Numbers for Partial Sums and a new MSLLN result for outer products of multivariate linear processes with LRD and HT. Experimental results are given in Section 5 and proof of main result (Theorem 1) is delayed until Section 6.

2. Example: Asymptotic Linear Observers by Adaptive Filtering. We refer to books Kushner and Yin [25], Ljung [27] and Soderstrom and Stoica [31] for standard vital applications in system identification, equalization, estimation, and adaptive control in stochastic systems. Rather than repeating these developments here, we just adapt a less-discussed, yet interesting application from Thanh, Yin and Wang [35]. They analyzed the convergence of double-indexed or triangular-array processes with mixing driving noises and random weights and established Marcinkiewicz Strong laws of large numbers and convergence rates for such problems (see Theorem 8 herein).

For motivation, Thanh, Yin and Wang considered the least square estimate of internal state of a multi-input-single-output linear-time-invariant system and showed that the estimation error is a special case of triangular-array processes with random

weights and mixing driving noise. Alternatively to least squares, the internal states of linear observers can be estimated through adaptive filtering algorithms.

Consider the following linear time-invariant system operating near steady state

(9)
$$\begin{cases} \dot{X}(t) = AX(t) + Bu(t), \\ Y(t) = CX(t), \end{cases}$$

where $A \in \mathbb{R}^{m_0 \times m_0}$, $B \in \mathbb{R}^{m_0 \times m_1}$ and $C \in \mathbb{R}^{1 \times m_0}$ are known system matrices. We are interested about estimating the state $X(t)$. However, since we are operating near steady state $X(t) \approx h$ for some unknown h. $X(t)$, i.e. h, must be estimated through output Y. $Y(t)$ is measured only at a sequence of irregular (i.e. random) sampling time instants $\{t_k\}$ with measured values y_{k+1} corrupted by correlated noise $\{d_k\}$:

(10)
$$y_{k+1} = Y(t_k) + d_k.$$

(Irregular sampling time sequences $Y(t_k)$ are sometimes generated actively by input control or threshold adaptation under binary-valued sensors, or passively due to event-triggered sampling or low-resolution signal quantization.) The goal is to estimate the state $X(t)$ from information on the control input $u(t)$, $\{t_k\}$, and $\{y_{k+1}\}$, all of which are known or learnt in real time. The internal state in (9) satisfies

(11)
$$X(t_{k+1}) = e^{A(t_{k+1}-t_k)}X(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau,$$

which we can not observe. Rather, we get access to the observations at the sampling time sequence $\{t_k, k = 1, ..., n\}$

$$y_{k+1} = Ce^{A(t_{k+1}-t_k)}X(t_k) + C\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau + d_k$$

(12)
$$= x_k h + \nu_k + d_k,$$

where $x_k = Ce^{A(t_{k+1}-t_k)}$ and $\nu_k = C\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau$ can be built in real time from known system matrices and the observed sampling times. Letting $\epsilon_k = d_k + \nu_k$, we find (12) is the same as (1). Hence, the stochastic approximation algorithm can be used to recursively find the value of h that minimizes the mean-square error $h \rightarrow E|y_{k+1} - x_k^T h|^2$. h gives us the estimate of internal state of multi-input-single-output linear-time-invariant system X. In addition, if the estimated steady state h agrees with the calculated steady state value of X from the model itself, this supports the model and choice of system matrices.

If X were not close to steady state initially but the variation of $X(t_k)$ is rarely large between consecutive samples, then one should use a constant gain adaptive algorithm (as in Kouritzin [23]) to start and then switch to the type considered herein once close to steady state.

3. Notation and Theoretical Result. In this section, we define our notation and provide our results.

$\lvert x \rvert$	is Euclidean distance of some $\mathbb{R}^d$-vector x.
$\lVert C \rVert$	is $\sup_{\lvert x \rvert = 1} \lvert Cx \rvert$ for any $\mathbb{R}^{n \times m}$-matrix C.
$\lvert\lvert\lvert A \rvert\rvert\rvert^2$	$\sum_{n=1}^{d} \sum_{o=1}^{d} (A^{(n,o)})^2$.
$A^{(n,o)}$	is the $(n,o)^{\text{th}}$ components of $A \in \mathbb{R}^{d \times d}$.
$\lfloor t \rfloor$	is $\max\{i \in \mathbb{N}_0 : i \le t\}$ for any $t \ge 0$.
$\lceil t \rceil$	is $\min\{i \in \mathbb{N}_0 : i \ge t\}$ for any $t \ge 0$.
$a_{i,k} \overset{i}{\ll} b_{i,k}$	means $\forall k, \exists c_k > 0$ not depending on i s.t. $\lvert a_{i,k} \rvert \le c_k \lvert b_{i,k} \rvert\ \forall i, k$.
$\prod_{l=p}^{q} B_l$	is $B_q B_{q-1} \cdots B_p$ if $q \ge p$ or I if $p > q, \forall B_l \in R^{d \times d}$.
$a \vee b$	is $\max\{a, b\}$.
$a \wedge b$	is $\min\{a, b\}$.

3.1. Main Results. We will prove our results in a completely deterministic manner and then apply these results to each path. Therefore, we assume that $\chi \in (0, 1)$, d is a positive integer, $\{\bar{A}_k\}_{k=1}^{\infty}$ is a symmetric, positive semidefinite $R^{d \times d}$ -valued sequence, $\{\bar{b}_k\}_{k=1}^{\infty}$ is a $\mathbb{R}^d$-valued sequence and $\{\bar{h}_k\}_{k=1}^{\infty}$ is a $\mathbb{R}^d$-valued sequence satisfying:

$$(13) \qquad \bar{h}_{k+1} = \bar{h}_k + \frac{1}{k^\chi}(\bar{b}_k - \bar{A}_k \bar{h}_k) \quad \text{for all} \quad k = 1, 2, \dots$$

Our first main result establishes rates of convergence:

THEOREM 1. *Let $\gamma \in [0, \chi)$, $h \in \mathbb{R}^d$ and A be symmetric and positive-definite.*
a) *If*

$$(14) \qquad \lim_{n \to \infty} \left\lVert \frac{1}{n^\chi} \sum_{k=1}^{n} (\bar{A}_k - A) \right\rVert = 0, \quad and \quad \lim_{n \to \infty} \left\lvert \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^{n} (\bar{b}_k - \bar{A}_k h) \right\rvert = 0,$$

then $n^\gamma \lvert \bar{h}_n - h \rvert \to 0$ a.s. as $n \to \infty$.

b) *Conversely,* $\displaystyle \lim_{n \to \infty} \left\lvert \frac{1}{n^\chi} \sum_{k=1}^{n} (\bar{b}_k - \bar{A}_k h) \right\rvert = 0$, *if* $\displaystyle \lim_{k \to \infty} \left\lvert k^{1-\chi}(\bar{h}_k - h) \right\rvert = 0$ *and*

$$(15) \qquad \frac{1}{n^\chi} \sum_{k=1}^{n} k^{\chi - 1} \lVert \bar{A}_k \rVert \quad \text{is bounded in } n.$$

Now we state the almost sure version of the above theorem as the following corollary:

COROLLARY 2. *Let $\gamma \in [0, \chi)$, $h \in \mathbb{R}^d$ and A be symmetric and positive-definite.*
a) *If*

$$(16)\ \lim_{n \to \infty} \left\lVert \frac{1}{n^\chi} \sum_{k=1}^{n} (A_k - A) \right\rVert = 0 \quad a.s.\ and \quad \lim_{n \to \infty} \left\lvert \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^{n} (b_k - A_k h) \right\rvert = 0 \quad a.s.,$$

then $n^\gamma \lvert h_n - h \rvert \to 0$ a.s. as $n \to \infty$.

b) *Conversely,* $\displaystyle \lim_{n \to \infty} \left\lvert \frac{1}{n^\chi} \sum_{k=1}^{n} (b_k - A_k h) \right\rvert = 0\,a.s.$, *if* $\displaystyle \lim_{k \to \infty} \left\lvert k^{1-\chi}(h_k - h) \right\rvert = 0\,a.s.$ *and*

$\displaystyle \frac{1}{n^\chi} \sum_{k=1}^{n} k^{\chi - 1} \lVert A_k \rVert$ *is bounded in n almost surely.*

Proof. **a)** Fix ω such that (16) is true, recall (6); set $\bar{A}_k = A_k(\omega)$, $\bar{b}_k = b_k(\omega)$ and $\bar{h}_k = h_k(\omega)$ for all k; and apply Theorem 1. **b)** is similar. $\square$

Corollary 2 implies $h_n(\omega)$, the solution of (2), converges to $h = A^{-1}b$ a.s.

REMARK 1. *Lemma 11 of Appendix establishes that the first equation of (14) implies (15).*

Indeed, to establish the rate of convergence $n^\gamma |\bar{h}_n - h| \to 0$ a.s., one need only check standard conditions for the MSLLN in (16), which is less onerous task than checking the technical conditions in Corollary 1 or Corollary 3 in [34] say. Indeed, there appears to be a need for some extra stability in [34] by the imposition that "the real parts of the eigenvalues of A should be strictly less than a certain negative value depending on the asymptotic properties of $\{\gamma_n\}$ and $\{\delta_n\}$". We do not need any such extra condition.

Generally, we do not know h when using stochastic approximation so we cannot just verify second condition in (14) (or (16)) but rather use the following corollary instead of Theorem 1 (or Corollary 2).

COROLLARY 3. *Suppose $\gamma \in [0, \chi)$ and A is a symmetric positive-definite matrix.*

$$(17) \qquad \frac{1}{n^{\chi-\gamma}} \sum_{k=1}^{n} (\bar{b}_k - b) \to 0 \ \text{ and } \ \frac{1}{n^{\chi-\gamma}} \sum_{k=1}^{n} (\bar{A}_k - A) \to 0 \ \text{ a.s.}$$

Then, $n^\gamma |\bar{h}_n - h| \to 0$ a.s. as $n \to \infty$.

Finally, we give a version of the theorem for linear processes under very general and verifiable conditions.

THEOREM 4. *Let $\{\Xi_l\}$ be i.i.d. zero-mean random $\mathbb{R}^m$-vectors such that*

$$\sup_{t \geq 0} t^\alpha P(|\Xi_1|^2 > t) < \infty \quad \text{for some } \alpha \in (1, 2)$$

$(C_l)_{l \in \mathbb{Z}}$ *be $\mathbb{R}^{(d+1) \times m}$-matrices such that $\sup_{l \in \mathbb{Z}} |l|^\sigma \|C_l\| < \infty$ for some $\sigma \in \left(\frac{1}{2}, 1\right]$,*

$$(x_k^T, y_{k+1})^T = \sum_{l=-\infty}^{\infty} C_{k-l} \Xi_l,$$

$A_k = x_k x_k^T$, $b_k = y_{k+1} x_k$ *and* $A = E[x_k x_k^T]$ *and* $b = E[y_{k+1} x_k]$.
Then, $n^\gamma |\bar{h}_n - h| \to 0$ a.s. as $n \to \infty$ a.s. for any $\gamma < \gamma_0(\chi) \doteq (\chi - \frac{1}{\alpha}) \wedge (\chi + 2\sigma - 2)$.

REMARK 2. *Theorem 4 follows from Corollary 3 and Theorem 9 (to follow), by letting $\frac{1}{p} = \chi - \gamma$ and $\overline{X}_k^T = X_k^T = (x_k^T, y_{k+1})$ and correspondingly, $\overline{\overline{\Xi}}_l = \Xi_l$, $\overline{C}_l = C_l$ and $\overline{\sigma} = \sigma$. σ and α are long-range dependence and heavy-tail parameters, respectively. Theorem 9 also appears in [20, Theorem 4].*

REMARK 3. *$\sup_{l \in \mathbb{Z}} |l|^{\sigma''} \|C_l\| < \infty$ for some $\sigma'' > 1$ would be the (normal) short-range dependence and this clearly implies our weaker $\sup_{l \in \mathbb{Z}} |l|^\sigma \|C_l\| < \infty$ condition for some $\sigma \in \left(\frac{1}{2}, 1\right]$, which allows for long-range dependence. Our tail condition $\sup_{t \geq 0} t^\alpha P(|\Xi_1|^2 > t) < \infty$ is implied by the moment condition $E|\Xi_1|^{2\alpha} < \infty$. However, it too is general enough to allow non-standard (heavy-tailed) $\{A_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$, since $\alpha < 2$ corresponds to moments less than 2 for A_k and b_k.*

4. Marcinkiewicz Strong Law of Large Numbers for Partial Sums. Our basic assumptions are MSLLN for random variables for $\{A_k\}$ and $\{b_k\}$. (Technically, our assumptions are even more general as they allow the non-MSLLN case where $\chi - \gamma \leq \frac{1}{2}$ that could be verified by some other method in some special situations.) The beauty of this MSLLN assumption is that: 1) It is minimal in the sense that the linear algorithm with $A_k = I$ and $\mu_k = \frac{1}{k}$ reduces to the partial sums $h_{k+1} - h = \frac{1}{k} \sum_{j=1}^{k} (b_j - b)$ (since $h = b$ when $A = I$) so a rate of convergence in the algorithm solution h_k implies a MSLLN for random variables $\{b_j\}$. 2) MSLLNs hold under very general conditions, including heavily-tailed and long-range dependent data. Hence, we review some of the literature in this area before giving simulation results for our theoretical work.

The classical independent case, due to Marcinkiewicz, is generalized slightly by Rio [29]:

THEOREM 5. *Let $\{X_i\}$ be an m-dependent, identically distributed sequence of zero-mean $\mathbb{R}$-valued random variables such that $E|X_1|^p < \infty$ for some $p \in (1, 2)$. Then,*

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} X_i \to 0 \ a.s.$$

Actually, Rio gives a more general m-dependent result on page 922 of his work. However, the important observation for us is that only the pth moment need be finite rather than a higher moment as is typical under some stronger dependence assumptions. Theorem 5 is quite useful in verifying our conditions when $\{A_k\}$ and $\{b_k\}$ may have heavily-tailed distributions but are independent or m-dependent. For example, if $\chi - \gamma \in \left(\frac{1}{2}, 1\right)$ and the $\{A_k\}$ and $\{b_k\}$ are defined as in (3) in terms of i.i.d. $\{x_k\}$ and $\{y_k\}$ with $E|x_1|^{\frac{2}{\chi - \gamma}} < \infty$, $E|y_1|^{\frac{2}{\chi - \gamma}} < \infty$, then $\{A_k\}_{k \geq M}$ and $\{b_k\}_{k \geq M}$ are identically distributed, M-dependent and

$$\frac{1}{n^{\chi - \gamma}} \sum_{k=1}^{n} (A_k - A) \to 0 \ \text{ and } \ \frac{1}{n^{\chi - \gamma}} \sum_{k=1}^{n} (b_k - b) \to 0$$

a.s., where $A = EA_k$ and $b = Eb_k$, by applying Theorem 5 for each component. Hence, (8) holds.

There are many other important results that include heavy-tails, long-range dependence or both. For example, Louhchi and Soulier [28] give the following result for linear symmetric α-stable (SαS) processes.

THEOREM 6. *Let $\{\zeta_j\}_{j \in \mathbb{Z}}$ be i.i.d. sequence of SαS random variables with $1 < \alpha < 2$ and $\{c_j\}_{j \in \mathbb{Z}}$ be a bounded collection such that $\sum_{j \in \mathbb{Z}} |c_j|^s < \infty$ for some $s \in [1, \alpha)$. Set $X_k = \sum_{j \in \mathbb{Z}} c_{k-j} \zeta_j$. Then, for $p \in (1, 2)$ satisfying $\frac{1}{p} > 1 - \frac{1}{s} + \frac{1}{\alpha}$*

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} X_i \to 0 \ a.s.$$

The condition $s < \alpha$ ensures $\sum_{j \in \mathbb{Z}} |c_j|^\alpha < \infty$ and thereby convergence of $\sum_{j \in \mathbb{Z}} c_{k-j} \zeta_j$. Moreover, $\{X_k\}$ not only exhibits heavy tails but also long-range dependence if, for example, $c_j = |j|^{-\sigma}$ for $j \neq 0$ and some $\sigma \in \left(\frac{1}{2}, 1\right)$. Notice there is interactions between the heavy tail condition and the long range dependent condition. In particular

for a given p, heavier tails (α becomes smaller) implies that you cannot have as long range dependence (s becomes smaller) and vice versa. Moreover, this result is difficult to apply in the stochastic approximation setting. For example, if wanted to apply it for $X_k = A_k$ in the scalar case, then we would need x_k such that $x_k^2 = A_k$ which is impossible when A_k is SαS.

One nice feature of mixing assumptions is that they usually transfer from random variables to functions (like squares) of random variables. There are many mixing results that handle long range dependence. For example, Berbee [2] gives a nice β-mixing result. However, strong mixing is one of the most general types of mixing that is more easily verified in practice. Hence, we will quote the following strong mixing result from Rio [29] (Theorem 1) in terms of the inverse $\alpha^{-1}(u) = \sup\{t \in \mathbb{R}^+ : \alpha_{\lfloor t \rfloor} > u\}$ of the strong mixing coefficients

$$\alpha_n = \sup_{k \in \mathbb{Z}} \sup_{A \in \sigma(X_i, i \leq k-n), B \in \sigma(X_k)} |P(AB) - P(A)P(B)|$$

and the complementary quantile function

$$Q_X(u) = \sup\{t \in \mathbb{R}^+ : P(|X| > t) > u\}.$$

THEOREM 7. *Let $\{X_i\}$ be an identically-distributed zero-mean sequence of $\mathbb{R}$-valued random variables such that $\int_0^1 [\alpha^{-1}(t/2)]^{p-1} Q_X^p(t) dt < \infty$ for some $p \in (1, 2)$. Then,*

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^n X_i \to 0 \ a.s.$$

Notice again that for a given p, heavier tails implies that you cannot have as long range dependence and vice versa: If you wanted to maintain the same value of the integral condition and there became more area under $P(|X| > t)$, then there would be more area under $Q_X^p(t)$ so the area under $[\alpha^{-1}(t/2)]^{p-1}$, which is equal to $2 \sum_{n=0}^{\infty} \alpha_n^{p-1}$, would have to decrease to compensate. Also, there can be difficulty in establishing that a given model satisfies the strong mixing condition with the required decay of mixing coefficients. Still, this is an important result for verifying our basic assumptions.

Another result in mixing area is given by Thanh, Yin and Wang [35, Theorem 3.11], whom considered MSLLN for double indexed and randomly weighted sums of mixing processes. Generally, they considered ρ^*-mixing types, which is defined as follows.

On the probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{A}$ and $\mathcal{B}$ be two sub-σ-algebras of $\mathcal{F}$. We denote by $\mathcal{L}_2(\mathcal{A})$ the space of all square integrable and $\mathcal{A}$-measurable random variables. The maximal coefficient of correlation is defined by

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathcal{L}_2(\mathcal{A}), g \in \mathcal{L}_2(\mathcal{B})} |corr(f, g)|.$$

Let $\{X_n, n \geq 1\}$ be a sequence of random variables. For a subset S of $\mathbb{N} = \{1, 2, ...\}$, $\sigma(S)$ means the σ-field generated by $\{X_n, n \in S\}$. For $n \geq 1$, define $\rho_n^* \doteq \rho^*(X, n) \doteq \sup \rho(\sigma(S), \sigma(T))$, where the supremum is taken over all pairs of nonempty finite sets S, T of $\mathbb{N}$ such that $dist(S, T) = \inf_{s \in S, t \in T} |s - t| \geq n$. The sequence $\{X_n, n \geq 1\}$ is said to be ρ^*-mixing if $\rho_n^* \to 0$ as $n \to \infty$.

THEOREM 8. *Let $0 \leq r < 1$, and let N be a positive integer. Let $1 \leq p < 2$, and let $\{X_n, n \geq 1\}$ be a sequence of mean zero strictly stationary random variables such that $\rho^*(X, N) \leq r$. Suppose that $\{A_{ni}, n \geq 1, 1 \leq i \leq n\}$ is an array of random variables such that, for each $n \geq 1$, the sequence $A_n = \{A_{ni}, 1 \leq i \leq n\}$ satisfies $\rho^*(A_n, N) \leq r$, and*

$$(18) \qquad \sum_{i=1}^{n} E(|A_{ni}|^q) = O(n) \qquad \text{for some} \quad q > \frac{2p}{2-p}.$$

If $E|X_1|^{2p} < \infty$, and $\{A_{ni}, n \geq 1, 1 \leq i \leq n\}$ is independent of $\{X_i, i \geq 1\}$, then

$$(19) \qquad \lim_{n \to \infty} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} A_{ni} X_i = 0 \quad a.s.$$

Notice that the above Theorem looked at the the moments of A_{ni} and X_i separately. However, if we look at the moment of products, we find by Hölder's inequality that $E|A_{ni}X_i|^{\frac{2p}{3-p}} < \infty$ and $\frac{2p}{3-p} \in [1, 4)$, so some heavy-tail situations are allowed. However, the ρ^*-mixing condition does not allow long-range dependence situations, since

$$\sum_{k=1}^{\infty} E[X_1 X_k] = \sum_{j=0}^{\infty} \sum_{k=1}^{N} E[X_1 X_{jN+k}] \leq N \sum_{j=0}^{\infty} r^j < \infty.$$

A new MSLLN for outer products of multivariate linear processes with long range dependence and heavy tails is studied in [20]. A new decoupling property is proved that shows the convergence rate is determined by the worst of the heavy tails or the long range dependence, but not the combination. This result used to obtain Marcinkiewicz Strong Law of Large Numbers for stochastic approximation (Theorem 4). The result is as follow.

THEOREM 9. *Let $\{\Xi_l\}$ and $\{\overline{\Xi}_l\}$ be i.i.d. zero mean random $\mathbb{R}^m$-vectors such that $\Xi_l = \left(\xi_l^{(1)}, ..., \xi_l^{(m)}\right)$, $\overline{\Xi}_l = \left(\overline{\xi}_l^{(1)}, ..., \overline{\xi}_l^{(m)}\right)$, $E[|\Xi_1|^2] < \infty$, $E[|\overline{\Xi}_1|^2] < \infty$ and $\max\limits_{1 \leq i,j \leq m} \sup\limits_{t \geq 0} t^\alpha P(|\xi_1^{(i)} \overline{\xi}_1^{(j)}| > t) < \infty$ for some $\alpha \in (1, 2)$. Moreover, suppose matrix sequences $(C_l)_{l \in \mathbb{Z}}, (\overline{C}_l)_{l \in \mathbb{Z}} \in \mathbb{R}^{(d+1) \times m}$ satisfy*

$$\sup_{l \in \mathbb{Z}} |l|^\sigma \|C_l\| < \infty, \ \sup_{l \in \mathbb{Z}} |l|^{\overline{\sigma}} \|\overline{C}_l\| < \infty \quad \text{for some} \quad (\sigma, \overline{\sigma}) \in \left(\frac{1}{2}, 1\right],$$

$X_k, \overline{X}_k$ take form of (4), $D_k = X_k \overline{X}_k^T$ and $D = E[X_1 \overline{X}_1^T]$. Then, for p satisfying $p < \frac{1}{2-\sigma-\overline{\sigma}} \wedge \alpha$

$$\lim_{n \to \infty} \frac{1}{n^{\frac{1}{p}}} \sum_{k=1}^{n} (D_k - D) = 0 \quad a.s.$$

REMARK 4. *This theorem actually shows the MSLLN for $D_k - E[D_k]$, where*
$$D_k = \begin{pmatrix} x_k x_k^T & y_{k+1} x_k \\ y_{k+1} x_k^T & y_{k+1}^2 \end{pmatrix}, \text{ which is more than required, so we can throw out}$$
the unneeded columns.

5. Experimental Results. In this section we now verify our results of the previous section experimentally in the stochastic approximation setting discussed in the introduction. In particular, we use *power law* or *folded t* distributions.

Power law distribution: A random variable ξ obeys a power law with parameters $\beta > 1$ and $x_{\min} > 0$, written $\xi \sim PL(x_{min}, \beta)$, if it has density

$$f(x) = \frac{\beta - 1}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\beta} \quad \forall\ x \geq x_{min}$$

Note that $E|\xi|^r = \begin{cases} x_{min}^r (\frac{\beta - 1}{\beta - 1 - r}) & r < \beta - 1 \\ \infty & r \geq \beta - 1 \end{cases}$.

Folded t distribution: A non-negative random variable ξ has a folded t distribution with parameter $\beta > 1$, written $\xi \sim Ft(\beta)$, if it has density

$$f(x) = \frac{2\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\beta - 1}{2})\sqrt{(\beta - 1)\pi}} \left(1 + \frac{x^2}{\beta - 1} \right)^{-\frac{\beta}{2}} \quad \forall\ x > 0.$$

Note that $E(|\xi|^r)$ exists if and only if $r < \beta - 1$.

Experimental results in this section are divided in two parts.

5.1. Heavy-tailed cases. Assume $N = 1$ in (3), dimension is $d = 2$ and $\{(x_k^{(1)}, x_k^{(2)}, \epsilon_k)^T,\ k = 1, 2, ...\}$ are i.i.d. random vectors so linear algorithm (2) reduces to:

$$(20) \qquad h_{k+1} = h_k + \mu_k(x_k y_{k+1} - x_k x_k^T h_k) = h_k + \mu_k(x_k x_k^T h + x_k \epsilon_k - x_k x_k^T h_k).$$

For consistency and performance, we always let $x_k^{(1)}, x_k^{(2)}$ and ϵ_k be independent. The runs are always initialized with $h_1 = (101, 101)^T$ and, for testing purposes, the optimal $h = (1, 1)^T$ is known.

EXAMPLE 1. *Let $x_k^{(1)}, x_k^{(2)} \sim PL(x_{min} = 1, \beta)$ and $\epsilon_k = \epsilon_k' - E(\epsilon_k')$ with $\epsilon_k' \sim PL(x_{min}' = 0.01, \beta)$. The normalized errors in 100 trial simulations, $\{h_n^{(i)}\}_{i=1}^{100}$, are averaged $\overline{rh} = \frac{1}{100} \sum_{i=1}^{100} \frac{|h_n^{(i)} - h|}{|h_1 - h|}$ and given in the Table 1 in terms of gain parameter χ, distributional parameter β and sample size n.*

Table 1: Algorithm performance-Power Law

$\chi \backslash \beta$	n=100000			n=750000			n=1500000		
	3.5	4	4.5	3.5	4	4.5	3.5	4	4.5
0.6	0.0864	0.0314	0.0169	0.0707	0.0243	0.0115	0.0548	0.0203	0.0099
0.7	0.0525	0.0190	0.0098	0.0487	0.0159	0.0067	0.0457	0.0141	0.0056
0.75	0.0397	0.0151	0.0082	0.0449	0.0137	0.0051	0.0456	0.0114	0.0042
0.8	0.0326	0.0136	0.0105	0.0448	0.0111	0.0038	0.0402	0.0087	0.0031
0.85	0.0314	0.0168	0.0549	0.0398	0.0085	0.0082	0.0324	0.0070	0.0035
0.9	0.0344	0.0719	0.2445	0.0438	0.0118	0.0764	0.0272	0.0079	0.0341
0.95	0.0902	0.3047	0.6631	0.3739	0.0897	0.3068	0.0248	0.0519	0.1963
0.98	0.2226	0.5733	1.0154	0.9219	0.2302	0.5251	0.0374	0.1488	0.3930
1	0.3876	0.8062	0.6631	1.1891	0.3745	0.6925	0.0662	0.2596	0.5644

The Marcinkiewicz threshold, $M = \frac{2}{(\beta - 1)}$, corresponding to $\beta = 3.5$, $\beta = 4$ and $\beta = 4.5$ are respectively $M = 0.8$, 0.67 and 0.57. Our theoretical results prove

convergence above this threshold. While the results in Table 1 are obviously still influenced by (heavy-tailed) randomness, one can see that convergence does appear to be taking place as one moves from $n = 100,000$ through $n = 750,000$ to $n = 1,500,000$ when $\chi > M$ and it is less clear that convergence is taking place when $\chi < M$. Furthermore, our (as well as prior) theoretical results predict rates of convergence that increase in χ. Indeed, in the case $\beta = 4$ our theoretical results suggest that $\chi \approx 1$ should result in a rate of convergence $n^{0.33}|h_n - h| \to 0$ a.s. while $\chi = 0.85$ should only result in a rate of convergence $n^{0.18}|h_n - h| \to 0$ a.s. Conversely, Table 1 demonstrates that $\chi = 0.85$ performs better, which seems to contradict the theory. However, this paradox is explained by the exploding constants discussion of the introduction and, in fact, points out that more refined theory, involving functional results, is needed. The proper way to use our theoretical results then is to predict the best χ (lowest value of $\overline{rh}$) in the range of $(M, 1]$ i.e. in $(0.8, 1]$, $(0.67, 1]$ and $(0.57, 1]$, respectively for our three β's.

Table 2: Best fixed χ-Power Law

	n=100000			n=750000			n=1500000		
β	3.5	4	4.5	3.5	4	4.5	3.5	4	4.5
Best χ	0.85	0.8	0.75	0.85	0.85	0.8	0.95	0.85	0.8
Resulting γ	0.05	0.13	0.18	0.05	0.18	0.23	0.15	0.18	0.23

The best χ's, corresponding to the smallest value of $\overline{rh}$ for $\beta = 3.5$, $\beta = 4$ and $\beta = 4.5$ and 3 different sample sizes, as well as the γ corresponding to the theoretical rate of convergence $o(n^{-\gamma})$ are summarized in Table 2. In all cases, the best value for χ is in the predicted range. As we explained, a faster decreasing gain is appropriate for a heavier-tailed distribution, which is also confirmed by Table 2. Notice also that the best χ increases in n, a phenomenon consistent with our exploding constants and the initial condition effect discussion. Now, we repeat the previous example with a different distribution. Since the results are consistent with those of the previous example, we will keep our discussion to a minimum.

EXAMPLE 2. *Let $x_k^{(1)}, x_k^{(2)} \sim Ft(\beta)$ and $\epsilon_k = \epsilon'_k - E(\epsilon'_k)$ with $\epsilon'_k \sim Ft(\beta)$. The simulation results for three β's: $3.5, 4$ and 4.5 with corresponding Marcinkiewicz thresholds, $M = \frac{2}{\beta-1}$, 0.8, 0.67 and 0.57 are given in Table 3 with sample sizes: $n = 50,000, 100,000$ and $750,000$.*

Table 3: Algorithm performance-Folded t

$\chi \backslash \beta$	n=50000			n=100000			n=750000		
	3.5	4	4.5	3.5	4	4.5	3.5	4	4.5
0.6	0.0958	0.0345	0.0221	0.0929	0.0336	0.0177	0.0590	0.0195	0.0104
0.7	0.0697	0.0245	0.0138	0.0661	0.0216	0.0112	0.0318	0.0120	0.0064
0.75	0.0599	0.0204	0.0113	0.0556	0.0173	0.0089	0.0336	0.0099	0.0050
0.8	0.0505	0.0172	0.0103	0.0439	0.0140	0.0075	0.0374	0.0076	0.0038
0.85	0.0399	0.0145	0.0098	0.0341	0.0118	0.0063	0.0339	0.0058	0.0029
0.9	0.0312	0.0133	0.0087	0.0278	0.0100	0.0057	0.0265	0.0048	0.0024
0.95	0.0275	0.0241	0.0097	0.0245	0.0089	0.0060	0.0205	0.0039	0.0021
0.98	0.0347	0.0475	0.0212	0.0274	0.0117	0.0121	0.0179	0.00371	0.0032
0.99	0.0404	0.0583	0.0295	0.0310	0.0149	0.0172	0.0173	0.00373	0.0048
1	0.0486	0.0700	0.0413	0.0369	0.0205	0.0249	0.0170	0.0039	0.0077

A summary of of best χ result is given in Table 4. Again, a smaller β corresponds to heavier tails and larger best χ. Moreover, as we predicted the best χ for $\beta = 3.5$, $\beta = 4$ and $\beta = 4.5$ in the range of $(0.8, 1]$, $(0.67, 1]$ and $(0.57, 1]$, respectively. Best χ's increase in sample size.

Table 4: Best fixed χ-Folded t

	n=50000			n=100000			n=750000		
β	3.5	4	4.5	3.5	4	4.5	3.5	4	4.5
Best χ	0.95	0.9	0.9	0.95	0.95	0.9	1	0.98	0.95
$\gamma <$	0.15	0.23	0.33	0.15	0.28	0.33	0.2	0.31	0.38

5.2. Combined Heavy-tailed and Long Range dependence case. If we take $N = 1$ and dimension $d = 1$, we have $(x_k, y_{k+1}) = \sum_{j=-\infty}^{\infty} C_{k-j}\Xi_j$, in which $C_j = (c_j, c_j)^T$ and $\Xi_j = (\xi_j^{(1)}, \xi_j^{(1)})_{j \in \mathbb{Z}}$ are i.i.d.. Hence, $x_k = \sum_{j \in \mathbb{Z}} c_{k-j}\xi_j^{(1)}$ and $y_{k+1} = \sum_{j \in \mathbb{Z}} c_{k-j}\xi_j^{(2)}$, where $\xi_j^{(2)} = h\xi_j^{(1)} + a_j$ and $\{a_j\}$'s are i.i.d. zero mean random variables. This relation between $\xi_j^{(1)}$ and $\xi_j^{(2)}$ is due to the fact that $y_{k+1} = x_k h + \epsilon_k$ and $\epsilon_k = \sum_{j \in \mathbb{Z}} c_{k-j}a_j$. We consider $\{c_j = |j|^{-\sigma}\}$, for $j \neq 0$ and $\sigma \in (\frac{1}{2}, 1]$, $c_0 = 1$. The linear algorithm (2) reduces to:

$$(21) \qquad h_{k+1} = h_k + \mu_k(x_k y_{k+1} - x_k^2 h_k) = h_k + \mu_k(x_k^2 h + x_k \epsilon_k - x_k^2 h_k).$$

The initial and optimal values are $h_1 = 401$ and $h = 1$.

EXAMPLE 3. *Let $\xi_j^{(1)} \sim PL(x_{min} = 0.01, \beta)$ and $a_j = f_j - E(f_j)$ with $f_j \sim PL(x'_{min} = 0.01, \beta)$. The simulation is done for one-sided process and since in computer we cannot technically do infinite sum, we assume summation over the range of $(0, 500,000)$. Similarly, the normalized errors in 100 trial simulations, $\{h_n^{(i)}\}_{i=1}^{100}$, are averaged and results for different χ's, β's and sample sizes n are presented in the following tables. The assumed σ is 0.65. The Marcinkiewicz threshold, $M = \frac{1}{\alpha} \vee (2 - 2\sigma)$, corresponding to $\beta = 4$, $\beta = 4.5$ and $\beta = 5$ is 0.7. Hence, predicted ranges for χ's with smallest $\overline{rh}$ will be $(0.7, 1]$. Simulation results are provided in Table 5 with summary of best χ in Table 6. It worth noticing that the convergence does not seem to take place below the Marcinkiewicz threshold and the best χs are in the predicted ranges and the normalized error decreases as β increases.*

Note that by considering $\sigma = 0.65$, the minimum of $2 - 2\sigma$ and $\frac{1}{\alpha}$ for all $\beta = 4, 4.5$ and 5 is $2 - 2\sigma$, hence we do not expect much change in the χ as β changes. In addition, the rate of convergence for all considered β's is determined by $\gamma < \chi + 2\sigma - 2$.

Table 5: Algorithm performance for LRD-HT cases with $\sigma = 0.65$

$\chi \backslash \beta$	n=100			n=5000			n=10,000		
	4	4.5	5	4	4.5	5	4	4.5	5
0.6	0.010917	0.006166	0.004508	0.013172	0.007826	0.005897	0.012465	0.007359	0.005527
0.7	0.000665	0.000237	0.000132	0.000958	0.000414	0.000262	0.000881	0.000377	0.000238
0.75	2.98e-05	7.88e-06	6.49e-06	9.77e-05	3.15e-05	1.70e-05	8.79e-05	2.83e-05	1.52e-05
0.8	1.02e-05	7.76e-06	6.39e-06	5.19e-06	3.80e-06	3.11e-06	4.72e-06	3.30e-06	2.69e-06
0.85	9.91e-06	7.77e-06	6.41e-06	5.01e-06	3.91e-06	3.21e-06	4.38e-06	3.37e-06	2.76e-06
0.9	9.93e-06	7.79e-06	6.45e-06	5.20e-06	4.12e-06	3.39e-06	4.54e-06	3.50e-06	2.86e-06
0.95	1.00e-05	7.90e-06	6.55e-06	5.63e-06	4.42e-06	3.62e-06	4.73e-06	3.65e-06	2.98e-06
0.98	1.01e-05	7.99e-06	6.61e-06	5.97e-06	4.69e-06	3.86e-06	4.88e-06	3.77e-06	3.08e-06
1	1.02e-05	8.04e-06	6.65e-06	6.28e-06	4.91e-06	4.03e-06	5.02e-06	3.89e-06	3.19e-06

Table 6: Best fixed χ-Power Law, LRD with $\sigma = 0.65$

	n=100			n=5000			n=10000		
β	4	4.5	5	4	4.5	5	4	4.5	5
Best χ	0.85	0.8	0.8	0.85	0.8	0.8	0.85	0.8	0.8
Resulting γ	0.15	0.1	0.1	0.15	0.1	0.1	0.15	0.1	0.1

6. The proof of Theorem 1. Part a) Step 1: Reduce rate of convergence to convergence of a transformed algorithm.

Letting $\eta_k = \left(\frac{k+1}{k}\right)^\gamma - 1$, setting $g_k = k^\gamma \left(\bar{h}_k - h\right)$ and using (13), one finds that

$$(22) \qquad g_{k+1} = g_k + \frac{1}{k^\chi}\left(\hat{b}_k - \hat{A}_k g_k\right) + \eta_k g_k,$$

where

$$(23) \qquad \hat{b}_k = (k+1)^\gamma \left(\bar{b}_k - \bar{A}_k h\right) \ \text{ and } \ \hat{A}_k = \left(\frac{k+1}{k}\right)^\gamma \bar{A}_k.$$

However, we have by Taylor's theorem and assumption that

$$(24) \qquad \frac{1}{n^\chi}\sum_{k=1}^n \eta_k \|A\| \le \frac{\gamma}{n^\chi}\sum_{k=1}^n k^{-1} \to 0, \quad \text{as } n \to \infty.$$

Step 2: Show MSLLN for new coefficients i.e. $\frac{1}{n^\chi}\sum_{k=1}^n \left(\hat{A}_k - A\right) \to 0$, and $\frac{1}{n^\chi}\sum_{k=1}^n \hat{b}_k \to 0$ as $n \to \infty$.

$$\left\| \frac{1}{n^\chi}\sum_{k=1}^n \left(\frac{k+1}{k}\right)^\gamma \left(\bar{A}_k - A\right) - \frac{2^\gamma}{n^\chi}\sum_{k=1}^n \left(\bar{A}_k - A\right) \right\|$$
$$= \left\| \frac{1}{n^\chi}\sum_{k=2}^n\sum_{j=2}^k \left[\left(\frac{j+1}{j}\right)^\gamma - \left(\frac{j}{j-1}\right)^\gamma\right]\left(\bar{A}_k - A\right) \right\|$$

$$\leq \sum_{j=2}^{n} \left[\left(\frac{j}{j-1}\right)^{\gamma} - \left(\frac{j+1}{j}\right)^{\gamma} \right] \frac{1}{n^{\chi}} \left(\left\| \sum_{k=2}^{n} (\bar{A}_k - A) \right\| + \left\| \sum_{k=2}^{j-1} (\bar{A}_k - A) \right\| \right)$$

$$\frac{1}{n^{\chi}} \left\| \sum_{k=2}^{n} (\bar{A}_k - A) \right\| \left[2^{\gamma} - \left(\frac{n+1}{n}\right)^{\gamma} \right]$$

$$+ \sum_{j=2}^{n} \left[\left(\frac{j}{j-1}\right)^{\gamma} - \left(\frac{j+1}{j}\right)^{\gamma} \right] \left(\frac{j-2}{n}\right)^{\chi} \frac{1}{(j-2)^{\chi}} \left\| \sum_{k=2}^{j-1} (\bar{A}_k - A) \right\|$$

which goes to zero by assumption and the Toeplitz lemma. By Taylor's theorem

$$\left| \frac{1}{n^{\chi}} \sum_{k=1}^{n} (k+1)^{\gamma} \left(\bar{b}_k - \bar{A}_k h\right) - \frac{1}{n^{\chi}(n+1)^{-\gamma}} \sum_{k=1}^{n} \left(\bar{b}_k - \bar{A}_k h\right) \right|$$

$$= \frac{1}{n^{\chi}} \left| \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \left[j^{\gamma} - (j+1)^{\gamma} \right] \left(\bar{b}_k - \bar{A}_k h\right) \right|$$

$$\leq \frac{1}{n^{\chi}} \sum_{j=2}^{n} \gamma j^{\gamma-1} \left| \sum_{k=1}^{j-1} \left(\bar{b}_k - \bar{A}_k h\right) \right|$$

(25) $$\leq \frac{\gamma}{n^{\chi}} \sum_{j=2}^{n} j^{\chi-1} \frac{1}{(j-1)^{\chi-\gamma}} \left| \sum_{k=1}^{j-1} \left(\bar{b}_k - \bar{A}_k h\right) \right|,$$

which goes to zero by the Toeplitz lemma.

Step 3: Convergence of g_k, hence the rate of convergence of $\bar{h}_k$ follows from the Proposition 10 with $b = 0$, $\hat{h}_k = g_k$, $h = 0$ and $\eta_k = \left(\frac{k+1}{k}\right)^{\gamma} - 1$. □

PROPOSITION 10. *Suppose* $\{\hat{A}_k\}_{k=1}^{\infty}$ *is a symmetric, positive-semidefinite* $R^{d \times d}$-*valued sequence;* A *is a (symmetric) positive-definite matrix;* $\chi \in (0,1)$; $\theta \in (\chi, 1]$; $\eta_k \leq \frac{\bar{\eta}}{k^{\theta}}$; $\bar{\eta} > 0$ *and*

(26) $$\hat{h}_{k+1} = \hat{h}_k + \frac{1}{k^{\chi}}(\hat{b}_k - \hat{A}_k \hat{h}_k) + \eta_k \hat{h}_k \quad \text{for all} \quad k = 1, 2, ...;$$

(27) $$\frac{1}{n^{\chi}} \sum_{k=1}^{n}(\hat{b}_k - b) \to 0 \text{ and } \frac{1}{n^{\chi}} \sum_{k=1}^{n}(\hat{A}_k - A) \to 0.$$

Then, $\hat{h}_n \to h \doteq A^{-1}b$ *as* $n \to \infty$.

Notation: To ease the notation in the sequel, we will take the product over no factors to be 1 and the sum of no terms to be 0. For convenience, we let:

(28) $$\nu_k := \hat{h}_k - h, \qquad Y_k := \hat{A}_k - A, \qquad z_k := \hat{b}_k - \hat{A}_k h.$$

Proof. **Step 1:** Show simplified algorithm with A_k's replaced converges.

We note $\dfrac{1}{n^{\chi}} \sum_{k=1}^{n} z_k \to 0$ and will show $\nu_k \to 0$, by proving $u_k \to 0$ and $w_k := \nu_k - u_k \to 0$, where

(29) $$u_{k+1} = \left(I - \frac{A}{k^{\chi}} + \eta_k I \right) u_k + \frac{z_k}{k^{\chi}} + \eta_k h \quad \text{subject to} \quad u_1 = \nu_1.$$

By induction, we have:

$$(30) \qquad u_n = \prod_{l=1}^{n-1} \left(I - \frac{A}{l^\chi} + \eta_l I \right) u_1 + \sum_{j=1}^{n-1} F_{j,n} z_j + \sum_{j=1}^{n-1} \bar{F}_{j,n} h \quad \text{for } n = 1, 2, \ldots$$

where

$$(31) \qquad \begin{cases} F_{j,n} = \frac{1}{j^\chi} \prod_{l=j+1}^{n-1} \left(I - \frac{A}{l^\chi} + \eta_l I \right) \\ \bar{F}_{j,n} = \eta_j j^\chi F_{j,n} \text{ for } j = 1, 2, \ldots, n-1, n = 2, 3, \ldots \end{cases}$$

Hence, by (30), (31) and Lemma 12 i, ii)

$$\lim_{n\to\infty} |u_n| \le \lim_{n\to\infty} \left\| \prod_{l=1}^{n-1} \left(I - \frac{A}{l^\chi} + \eta_l I \right) \right\| |u_1|$$

$$(32) \qquad + \lim_{n\to\infty} \left| \sum_{j=1}^{n-1} F_{j,n} z_j \right| + \lim_{n\to\infty} \left| \sum_{j=1}^{n-1} \bar{F}_{j,n} h \right| = 0.$$

Step 2: Transfer stability from A to blocks of A_k.
Define the blocks

$$(33) \qquad \begin{cases} n_k = \lfloor (ak)^{\frac{1}{1-\chi}} \rfloor := \max\{ i \in N_0 : i \le (ak)^{\frac{1}{1-\chi}} \} \\ I_k = \{ n_k, n_k + 1, \cdots, n_{k+1} - 1 \} \end{cases}$$

for $k = 0, 1, 2, \ldots$ and the block products

$$(34) \qquad U_k = \prod_{l \in I_k} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) \text{ and } V_{j,k} = \prod_{l=j+1}^{n_{k+1}-1} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) \frac{1}{j^\chi} Y_j.$$

For the U_k's we have

$$\prod_{l \in I_k} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) = I - \sum_{l \in I_k} \frac{\hat{A}_l}{l^\chi} + \sum_{l \in I_k} \eta_l I + \sum_{\substack{l_1, l_2 \in I_k \\ l_1 > l_2}} \left(\frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right) \left(\frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right)$$

$$\sum_{\substack{l_1, l_2, l_3 \in I_k \\ l_1 > l_2 > l_3}} \left(\frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right) \left(\frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right) \left(\frac{\hat{A}_{l_3}}{l_3^\chi} - \eta_{l_3} I \right) + \cdots (-1)^k \prod_{l \in I_k} \left(\frac{\hat{A}_l}{l^\chi} - \eta_l I \right)$$

so

$$\|U_k\| \le \left\| I - \sum_{l \in I_k} \frac{\hat{A}_l}{l^\chi} \right\| + \sum_{l \in I_k} \eta_l + \left\| \sum_{\substack{l_1, l_2 \in I_k \\ l_1 > l_2}} \left(\frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right) \left(\frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right) \right\|$$

$$+ \left\| \sum_{\substack{l_1, l_2, l_3 \in I_k \\ l_1 > l_2 > l_3}} \left(\frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right) \left(\frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right) \left(\frac{\hat{A}_{l_3}}{l_3^\chi} - \eta_{l_3} I \right) \right\|$$

$$(35) \qquad + \cdots + \prod_{l \in I_k} \left\| \frac{\hat{A}_l}{l^\chi} - \eta_l I \right\|.$$

However, we know that $\sum_{j_1 > j_2 > \cdots > j_k} a_{j_1} a_{j_2} \cdots a_{j_k} \leq \frac{1}{k!} \left(\sum_j a_j \right)^k$ for $a_j \geq 0$ so, it follows that

$$
\sum_{\substack{l_1, l_2 \in I_k \\ l_1 > l_2}} \left\| \frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right\| \left\| \frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right\|
$$

$$
+ \sum_{\substack{l_1, l_2, l_3 \in I_k \\ l_1 > l_2 > l_3}} \left\| \frac{\hat{A}_{l_1}}{l_1^\chi} - \eta_{l_1} I \right\| \left\| \frac{\hat{A}_{l_2}}{l_2^\chi} - \eta_{l_2} I \right\| \left\| \frac{\hat{A}_{l_3}}{l_3^\chi} - \eta_{l_3} I \right\| + \cdots + \prod_{l \in I_k} \left\| \frac{\hat{A}_l}{l^\chi} - \eta_l I \right\|
$$

$$
\leq \sum_{m=2}^{n_{k+1} - n_k} \frac{\left(\sum_{l \in I_k} \left(\frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \right)^m}{m!}.
$$

As a result, we find by (35) that

$$
\|U_k\| \leq \left\| I - A \sum_{l \in I_k} \frac{1}{l^\chi} \right\| + \left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| + \sum_{l \in I_k} \eta_l
$$

$$
\text{(36)} \qquad + \sum_{m=2}^{n_{k+1} - n_k} \frac{\left(\sum_{l \in I_k} \left(\frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \right)^m}{m!}.
$$

Now, let λ_{min} and λ_{max} be the smallest and biggest eigenvalues of A and define $a' = \frac{a}{1-\chi}$, where $a > 0$ is chosen small enough that

$$
\text{(37)} \qquad a' \leq \left\{ \frac{2}{\lambda_{min} + \|A\|}, \frac{1}{d\|A\|}, \frac{\lambda_{min}}{e^1 (d\|A\|)^2} \right\}.
$$

Then, by (33) and the fact that

$$
\frac{1}{1-\chi} (n_{k+1}^{1-\chi} - n_k^{1-\chi}) \leq \sum_{l \in I_k} \frac{1}{l^\chi} \leq \frac{1}{1-\chi} ((n_{k+1} - 1)^{1-\chi} - (n_k - 1)^{1-\chi})
$$

we have $\lim_{k \to \infty} \left(\sum_{l \in I_k} \frac{1}{l^\chi} - a' \right)$ is in the range of

$$
\left(\lim_{k \to \infty} \frac{n_{k+1}^{1-\chi} - n_k^{1-\chi} - a}{1-\chi}, \ \lim_{k \to \infty} \frac{n_{k+1}^{1-\chi} - n_k^{1-\chi} - a}{1-\chi} + \frac{n_k^{1-\chi} - (n_k - 1)^{1-\chi}}{1-\chi} \right)
$$

so by Taylor's theorem

$$
\lim_{k \to \infty} \left| \sum_{l \in I_k} \frac{1}{l^\chi} - a' \right| \leq \lim_{k \to \infty} \left\{ \frac{1}{1-\chi} \left| n_{k+1}^{1-\chi} - n_k^{1-\chi} - a \right| + \frac{1}{(n_k - 1)^\chi} \right\}
$$

$$
\text{(38)} \qquad\qquad = 0,
$$

which also implies

$$
\text{(39)} \qquad \lim_{k \to \infty} \sum_{l \in I_k} \eta_l \leq \bar{\eta} \lim_{k \to \infty} n_k^{\chi - \theta} \sum_{l \in I_k} \frac{1}{l^\chi} = 0.
$$

For arbitrary $\epsilon > 0$ one finds some $K_\epsilon > 0$ by (38) and (37) such that

$$\left\| I - A \sum_{l \in I_k} \frac{1}{l^\chi} \right\| = \max \left\{ \|A\| \sum_{l \in I_k} \frac{1}{l^\chi} - 1, 1 - \lambda_{min} \sum_{l \in I_k} \frac{1}{l^\chi} \right\}$$

(40) $$\leq 1 - \lambda_{min} a' + \epsilon \qquad \text{for all} \quad k \geq K_\epsilon$$

Moreover, we can use Lemma 13 of Appendix, (28), (27), (38), (39), Taylor's theorem and the fact $d\|A\|a' < 1$ and to obtain a $K'_\epsilon \geq K_\epsilon$ such that

$$\sum_{m=2}^{n_{k+1}-n_k} \frac{\left(\sum_{l \in I_k} \left(\frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \right)^m}{m!} \leq \sum_{m=2}^{n_{k+1}-n_k} \frac{\left(d\|A\| \sum_{l \in I_k} \frac{1}{l^\chi} + d\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \| + \sum_{l \in I_k} \eta_l \right)^m}{m!}$$

(41) $$\leq e^{1+3\epsilon} \frac{(d\|A\|a' + 3\epsilon)^2}{2} \qquad \text{for all } k \geq K'_\epsilon$$

Therefore, by (40), Lemma 12 iii), (36) and (41) one finds

$$\|U_k\| \leq \left\| I - A \sum_{l \in I_k} \frac{1}{l^\chi} \right\| + \sum_{l \in I_k} \eta_l + \left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| + \sum_{m=2}^{n_{k+1}-n_k} \frac{\left(\sum_{l \in I_k} \left(\frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \right)^m}{m!}$$

(42) $$\leq 1 - \lambda_{min} a' + 3\epsilon + e^{1+3\epsilon} \frac{(d\|A\|a' + 3\epsilon)^2}{2} \qquad \forall \, k \geq K'_\epsilon$$

Furthermore, using the fact that $a' < \frac{\lambda_{min}}{e^1(d\|A\|)^2}$ and making for $\epsilon > 0$ small enough, we find from (42) that, there exists a $0 < \gamma < 1$ and an integer $k_1 > 0$ such that

(43) $$\|U_k\| \leq \gamma \qquad \text{for all } k \geq k_1$$

Step 3: Convergence of remainder w_n along a subsequence using block stability of A_k.
By (26), (28), (30) and $w_k := \nu_k - u_k \to 0$

(44) $$w_{n+1} = \left(I - \frac{\hat{A}_n}{n^\chi} + \eta_n I \right) w_n - \frac{1}{n^\chi} Y_n u_n \quad \text{for } n = 1, 2, \cdots$$

so it follows by (44) that

$$w_n = \prod_{l=n_k}^{n-1} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) w_{n_k}$$

(45) $$- \sum_{j=n_k}^{n-1} \prod_{l=j+1}^{n-1} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) \frac{Y_j u_j}{j^\chi} \quad \forall \quad n \geq n_k.$$

In particular,

(46) $$w_{n_{k+1}} = U_k w_{n_k} - \sum_{j \in I_k} V_{j,k} u_j \quad \text{for} \quad k = 0, 1, \cdots,$$

where U_k is defined in (34) and

$$(47) \qquad V_{j,k} = \prod_{l=j+1}^{n_{k+1}-1} \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) \frac{1}{j^\chi} Y_j.$$

By Lemma 12 v) and (47) we obtain,

$$\|V_{j,k}\| \leq \prod_{l=j+1}^{n_{k+1}-1} \left\| \left(I - \frac{\hat{A}_l}{l^\chi} + \eta_l I \right) \right\| \frac{\|Y_j\|}{j^\chi}$$

$$(48) \qquad \leq \prod_{l \in I_k} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \frac{\|Y_j\|}{j^\chi} \stackrel{j,k}{\ll} \frac{\|Y_j\|}{j^\chi} \quad \text{for } j \in I_k, \ k = 0, 1, \ldots$$

Therefore, by (43), (48), (34), (46), and (15) we have

$$(49) \qquad |w_{n_k}| \stackrel{k}{\ll} \gamma^{k-k_1} |w_{n_{k_1}}| + \sum_{l=k_1}^{k-1} \gamma^{k-l-1} \sum_{j \in I_l} \frac{\|A\| + \|\hat{A}_j\|}{j^\chi} |u_j| \quad \forall k \geq k_1.$$

In addition,

$$\sum_{j \in I_l} \frac{\|A\| + \|\hat{A}_j\|}{j^\chi} |u_j| = \|A\| \sum_{j \in I_l} \frac{1}{j^\chi} |u_j| + \sum_{j \in I_l} \frac{\|\hat{A}_j\|}{j^\chi} |u_j|$$

so using Lemma 12 iv), (32), (38) and finally applying Toeplitz Lemma, we obtain

$$(50) \qquad \lim_{l \to \infty} \sum_{j \in I_l} \frac{\|A\| + \|\hat{A}_j\|}{j^\chi} |u_j| = 0.$$

Moreover, since

$$(51) \qquad \sum_{l=k_1}^{k-1} \gamma^{k-l-1} = \frac{1 - \gamma^{k-k_1}}{1 - \gamma} \stackrel{k}{\ll} 1 \qquad \text{for all } k = k_1, k_1 + 1, \cdots$$

it follows from (49), (50), (51) and the Toeplitz Lemma with $a_{l,k} = \gamma^{k-l-1} 1_{k_1 \leq l \leq k-1}$ and $x_l = \sum_{j \in I_l} \frac{\|A\| + \|\hat{A}_j\|}{j^\chi} |u_j|$ that

$$\lim_{k \to \infty} |w_{n_k}| \leq \lim_{k \to \infty} \gamma^{k-k_1} |w_{n_{k_1}}|$$

$$(52) \qquad + \lim_{k \to \infty} \sum_{l=k_1}^{k-1} \gamma^{k-l-1} \sum_{j \in I_l} \frac{\|A\| + \|\hat{A}_j\|}{j^\chi} |u_j| = 0.$$

Step 4: Use $w_{n_k} \to 0$ to show block convergence $\max_{n \in I_k} |w_n| \to 0$.
Now, we return to (45) and find for $n \in I_k$

$$|w_n| \leq \prod_{l=n_k}^{n-1} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) |w_{n_k}| + \sum_{j=n_k}^{n-1} \prod_{l=n_k}^{n-1} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \frac{\|Y_j\|}{j^\chi} |u_j|$$

$$\leq \prod_{l \in I_k} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \left\{ |w_{n_k}| + \sum_{j \in I_k} \frac{\|Y_j\|}{j^\chi} |u_j| \right\}$$

$$(53) \qquad \leq \prod_{l \in I_k} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \left\{ |w_{n_k}| + \sum_{j \in I_k} \frac{\|\hat{A}_j\| + \|A\|}{j^\chi} |u_j| \right\}.$$

Finally, by (53), (52), Lemma 12 v), and (50) we obtain

$$(54) \qquad \lim_{k\to\infty} \max_{n\in I_k} |\,w_n\,| = 0. \qquad \square$$

Part b) By (13) and (28), $z_k = k^\chi(\nu_{k+1} - \nu_k) + \bar{A}_k\nu_k$. Averaging, then reordering the sum, we have

$$\frac{1}{n^\chi} \sum_{k=1}^{n} z_k = \frac{1}{n^\chi}\left(\sum_{k=1}^{n} k^\chi(\nu_{k+1} - \nu_k) + \sum_{k=1}^{n} \bar{A}_k\nu_k \right)$$

$$= \nu_{n+1} - \frac{1}{n^\chi} \sum_{k=1}^{n} (k^\chi - (k-1)^\chi)\nu_k + \frac{1}{n^\chi} \sum_{k=1}^{n} \bar{A}_k\nu_k$$

so

$$\left| \frac{1}{n^\chi} \sum_{k=1}^{n} z_k \right| \leq |\,\nu_{n+1}\,| + \sum_{k=1}^{n} \frac{k^\chi - (k-1)^\chi}{n^\chi} |\,\nu_k\,|$$

$$(55) \qquad\qquad + \sum_{k=1}^{n} \frac{k^{\chi-1}}{n^\chi} \|\bar{A}_k\| k^{1-\chi} |\,\nu_k\,|.$$

The second and third terms on the RHS of (55) converge to 0 by the Toeplitz lemma with $a_{n,k} = \frac{k^\chi - (k-1)^\chi}{n^\chi}$, $x_k = |\nu_k|$ and with $a_{n,k} = \frac{k^{\chi-1}\|\bar{A}_k\|}{n^\chi}$, $x_k = k^{\chi-1}|\nu_k|$ respectively. $\square$

7. Appendix. We first establish our promised comparison on our conditions.

LEMMA 11. $\lim\sup\limits_{n\to\infty} \left\| \frac{1}{n^\chi} \sum\limits_{k=1}^{n} (\bar{A}_k - A) \right\| = 0$ *implies* $\frac{1}{n^\chi} \sum\limits_{k=1}^{n} k^{\chi-1}\|\bar{A}_k\|$ *is bounded in* n.

Proof. By Lemma 13 (to follow) and the fact that $\sum\limits_{k=1}^{n} k^{\chi-1} \leq \frac{n^\chi}{\chi}$, one finds that

$$\frac{1}{n^\chi} \sum_{k=1}^{n} k^{\chi-1}\|\bar{A}_k\| \leq \frac{d}{n^\chi} \left\| \sum_{k=1}^{n} k^{\chi-1}\bar{A}_k \right\|$$

$$\leq \frac{d}{n^\chi} \left\| \sum_{k=1}^{n} k^{\chi-1}(\bar{A}_k - A) \right\| + \frac{d}{n^\chi} \|A\| \sum_{k=1}^{n} k^{\chi-1}$$

$$(56) \qquad\qquad \leq \frac{d}{n^\chi} \left\| \sum_{k=1}^{n} k^{\chi-1}(\bar{A}_k - A) \right\| + \frac{d\|A\|}{\chi}.$$

Noting $\sum\limits_{j=2}^{k}(j^{\chi-1} - (j-1)^{\chi-1}) = k^{\chi-1} - 1$, setting $C = \frac{d\|A\|}{\chi} + \sup\limits_n \frac{d}{n^\chi}\left\|\sum\limits_{k=1}^{n}(\bar{A}_k - A)\right\| < \infty$ and interchanging summation order, we have

$$\frac{1}{n^\chi} \sum_{k=1}^{n} k^{\chi-1}\|\bar{A}_k\| \leq \frac{d}{n^\chi} \left\| \sum_{k=2}^{n}\sum_{j=2}^{k}(j^{\chi-1} - (j-1)^{\chi-1})(\bar{A}_k - A) \right\| + C$$

$$(57) \qquad\qquad \leq \frac{d}{n^\chi} \left\| \sum_{j=2}^{n}(j^{\chi-1} - (j-1)^{\chi-1})\sum_{k=j}^{n}(\bar{A}_k - A) \right\| + C.$$

However, by Taylor's theorem $(j^{1-\chi} - (j-1)^{1-\chi}) \le (1-\chi)(j-1)^{-\chi}$, so by (57) we have

$$\frac{1}{n^\chi} \sum_{k=1}^n k^{\chi-1} \|\bar{A}_k\|$$

$$\le d \sum_{j=2}^n \frac{(j^{1-\chi} - (j-1)^{1-\chi})}{j^{1-\chi}(j-1)^{1-\chi}} \cdot \frac{1}{n^\chi} \left\| \sum_{k=j}^n (\bar{A}_k - A) \right\| + C$$

$$\le d \sum_{j=2}^n \frac{(j-1)^{-\chi}(1-\chi)}{j^{1-\chi}(j-1)^{1-\chi}} \cdot \frac{1}{n^\chi} \left(\left\| \sum_{k=1}^n (\bar{A}_k - A) \right\| + \left\| \sum_{k=1}^{j-1} (\bar{A}_k - A) \right\| \right) + C$$

$$\le 2d(1-\chi) \sum_{j=2}^n \frac{1}{j^{2-\chi}} \left(\left\| \frac{1}{n^\chi} \sum_{k=1}^n (\bar{A}_k - A) \right\| + \left\| \frac{1}{(j-1)^\chi} \sum_{k=1}^{j-1} (\bar{A}_k - A) \right\| \right) + C$$

This final term is bounded by the Toeplitz lemma and our hypothesis. $\square$

We give our list of technical bounds used in the proof of Proposition 10.

LEMMA 12. *Assume the setting of Proposition 10; and $F_{j,k}$, I_k, $\{z_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ are as defined in (33), (31) and (28). Then, following are true:*

i) $\displaystyle \lim_{n \to \infty} \left\| \prod_{l=1}^{n-1} \left(I - \frac{A}{l^\chi} + \eta_l I \right) \right\| = 0$

ii) $\displaystyle \lim_{n \to \infty} \left| \sum_{j=1}^{n-1} F_{j,n} z_j \right| = 0$ *and* $\displaystyle \lim_{n \to \infty} \left| \sum_{j=1}^{n-1} \bar{F}_{j,n} h \right| = 0$

iii) $\displaystyle \lim_{k \to \infty} \left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| = 0$

iv) $\displaystyle \sum_{l \in I_k} \left(\frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right)^k \ll 1$ *for all $k = 0, 1, \cdots$*

v) $\displaystyle \prod_{l \in I_k} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right)^k \ll 1$ *for all $k = 0, 1, \cdots$*

Proof. **i)** We know $\left\| I - \frac{A}{l^\chi} + \eta_l I \right\|$ is the maximum eigenvalue of $\left((1 + \eta_l)I - \frac{A}{l^\chi} \right)$ and

$$0 \le \left\| \prod_{l=1}^{n-1} \left((1+\eta_l)I - \frac{A}{l^\chi} \right) \right\| \le \prod_{l=1}^{n-1} \left\| (1+\eta_l)I - \frac{A}{l^\chi} \right\|.$$

Let $\lambda_{min} > 0$ be the minimum eigenvalue of A; recall from the statement of Proposition 10 that $\eta_k \le \frac{\bar{\eta}}{k^\theta}$ and $\theta < \chi$; and fix l^* large enough that: $1 + \eta_l - \frac{\lambda_{min}}{l^\chi} > 0 \, \forall \, l > l^*$.

Using the fact that $\prod_l (1 + x_l) \le \exp \left(\sum_l x_l \right)$, one finds

$$\prod_{l=l^*}^{n-1} \left\| (1+\eta_l)I - \frac{A}{l^\chi} \right\| \le \prod_{l=l^*}^{n-1} \left(1 + \frac{\bar{\eta}}{l^\theta} - \frac{\lambda_{min}}{l^\chi} \right)$$

$$\le \exp \left(\int_{l^*-1}^{n-1} \frac{\bar{\eta}}{x^\theta} dx - \int_{l^*}^n \frac{\lambda_{min}}{x^\chi} dx \right)$$

$$\leq \exp\left(D + \frac{\bar{\eta}}{1-\theta}(n-1)^{1-\theta} - \frac{\lambda_{min}}{1-\chi}n^{1-\chi}\right)$$

$$\overset{n}{\ll} \exp\left(\frac{-\lambda_{min}}{2-2\chi}n^{1-\chi}\right)$$

for some $D \in \mathbb{R}$. Hence,

$$(58) \qquad \prod_{l=l^*}^{n-1}\left\|(1+\eta_l)I - \frac{A}{l^\chi}\right\| \to 0 \text{ as } n \to \infty.$$

ii) $\|(r^\chi + \eta_r r^\chi - (r-1)^\chi)I - A\| \leq |(r^\chi - (r-1)^\chi)| + \bar{\eta}r^{\chi-\theta} + \|A\| \leq 1 + \bar{\eta} + \|A\|$
is upper bounded $\forall r > 1$ since $\chi \in (0,1)$. Hence, by (31) we have

$$\|F_{r-1,n} - F_{r,n}\| = \left\|\prod_{l=r+1}^{n-1}\left((1+\eta_l)I - \frac{A}{l^\chi}\right)\left[\frac{1}{(r-1)^\chi}\left((1+\eta_r)I - \frac{A}{r^\chi}\right) - \frac{1}{r^\chi}I\right]\right\|$$

$$\leq \left\|\prod_{l=r+1}^{n-1}\left((1+\eta_l)I - \frac{A}{l^\chi}\right)\right\|\frac{1}{r^\chi(r-1)^\chi}$$

$$\times \|(r^\chi + \eta_r r^\chi - (r-1)^\chi)I - A\|$$

$$(59) \qquad \overset{r,n}{\ll} \frac{1}{r^\chi(r-1)^\chi}\left\|\prod_{l=r+1}^{n-1}\left((1+\eta_l)I - \frac{A}{l^\chi}\right)\right\|$$

for all $r = 2,3,...,n-1$, $n = 3,4,....$ Letting λ denote an arbitrary eigenvalue of A, setting $L^c = \{l : \frac{\lambda}{l^\chi} - 1 - \eta_l \geq c\}$, noting that $\left\|\left(1 + \eta_l - \frac{\lambda}{l^\chi}\right)I\right\| \leq \left(\frac{\lambda}{l^\chi} - 1 - \eta_l\right) \vee \exp\left(\eta_l - \frac{\lambda}{l^\chi}\right)$ and defining constant $C \doteq \prod_{l \in L^1}\left(\frac{\lambda}{l^\chi} - 1 - \eta_l\right) \times \exp\left(\sum_{l \in L^0}\frac{\lambda}{l^\chi} - \eta_l\right)$ we have that

$$\left\|\prod_{l=r+1}^{n-1}\left(1 + \eta_l - \frac{\lambda}{l^\chi}\right)I\right\| \leq \prod_{l \in L^1}\left(\frac{\lambda}{l^\chi} - 1 - \eta_l\right) \times \exp\left(\sum_{l=r+1,l\notin L^0}^{n-1}\eta_l - \frac{\lambda}{l^\chi}\right)$$

$$\leq C\exp\left(\sum_{l=r+1}^{n-1}\frac{\bar{\eta}}{l^\theta} - \frac{\lambda}{l^\chi}\right)$$

$$\overset{r,n}{\ll} \exp\left(-\frac{\lambda_{min}}{2-2\chi}\{n^{1-\chi} - (r+1)^{1-\chi}\}\right)$$

and it follows from (60), the fact that the eigenvectors of A span $\mathbb{R}^d$ and the principle of uniform boundedness that

$$(60) \qquad \left\|\prod_{l=r+1}^{n-1}\left((1+\eta_l)I - \frac{A}{l^\chi}\right)\right\| \overset{r,n}{\ll} e^{-\frac{\lambda_{min}}{2-2\chi}\{n^{1-\chi}-(r+1)^{1-\chi}\}}.$$

It follows by (31), (59) and (60) that

$$\sum_{r=2}^{n-1}(r-1)^\chi\|F_{r-1,n} - F_{r,n}\| \overset{n}{\ll} \sum_{r=2}^{n-1}\frac{1}{r^\chi}e^{-\frac{\lambda_{min}}{2-2\chi}\{n^{1-\chi}-(r+1)^{1-\chi}\}}$$

$$\overset{n}{\ll} e^{-\frac{\lambda_{min}}{2-2\chi}n^{1-\chi}}\int_2^n \frac{1}{t^\chi}e^{\frac{\lambda_{min}}{2-2\chi}t^{1-\chi}}dt$$

$$(61) \qquad\qquad\qquad \overset{n}{\ll} 1 \quad \forall n = 3,4,...$$

Next, $\sum_{j=1}^{n-1} F_{j,n} z_j = \sum_{j=1}^{n-1} F_{n-1,n} z_j + \sum_{j=1}^{n-1} \left(\sum_{r=j+1}^{n-1} F_{r-1,n} - F_{r,n} \right) z_j$ and

$$\left| \sum_{j=1}^{n-1} \sum_{r=j+1}^{n-1} (F_{r-1,n} - F_{r,n}) z_j \right| \leq \sum_{r=2}^{n-1} \|F_{r-1,n} - F_{r,n}\| \left| \sum_{j=1}^{r-1} z_j \right|.$$

Therefore, by assumption, (31), (61) and Toeplitz' lemma with $x_r = \frac{1}{(r-1)^\chi} \left| \sum_{j=1}^{r-1} z_j \right|$ and $a_{n,r} = (r-1)^\chi \|F_{r-1,n} - F_{r,n}\|$ we have:

$$\left| \sum_{j=1}^{n-1} F_{j,n} z_j \right| \leq \|F_{n-1,n}\| \left| \sum_{j=1}^{n-1} z_j \right| + \left| \sum_{j=1}^{n-1} \sum_{r=j+1}^{n-1} (F_{r-1,n} - F_{r,n}) z_j \right|$$

$$(62) \qquad \leq \frac{1}{(n-1)^\chi} \left| \sum_{j=1}^{n-1} z_j \right| + \sum_{r=2}^{n-1} \|F_{r-1,n} - F_{r,n}\| \left| \sum_{j=1}^{r-1} z_j \right| \to 0.$$

as $n \to \infty$. Turning to the second limit in ii), we have by (31) and (60) that

$$(63) \qquad \sum_{j=1}^{n-1} \|\bar{F}_{j,n}\| \stackrel{n}{\ll} \sum_{j=1}^{n-1} j^{-\chi} e^{-\frac{\lambda_{min}}{2-2\chi} \{n^{1-\chi} - (j+1)^{1-\chi}\}} j^{\chi-\theta}.$$

However,

$$\sum_{j=1}^{n-1} \frac{1}{j^\chi} e^{-\frac{\lambda_{min}}{2-2\chi} \{n^{1-\chi} - (j+1)^{1-\chi}\}} \stackrel{n}{\ll} e^{-\frac{\lambda_{min}}{2-2\chi} n^{1-\chi}} \int_1^n \frac{1}{t^\chi} e^{\frac{\lambda_{min}}{2-2\chi} t^{1-\chi}} dt$$

$$(64) \qquad \qquad \stackrel{n}{\ll} 1$$

for all n so the second limit in ii) follows by the Toeplitz lemma.

iii) Since $\frac{1}{l^\chi} = \frac{1}{n_k^\chi} + \sum_{r=n_k}^{l-1} \left(\frac{1}{(r+1)^\chi} - \frac{1}{r^\chi} \right) \qquad \forall \quad l \in I_k$, one has that

$$\left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| \leq \frac{1}{n_k^\chi} \left\| \sum_{l \in I_k} Y_l \right\| + \left\| \sum_{\substack{r < l \\ r, l \in I_k}} \left(\frac{1}{(r+1)^\chi} - \frac{1}{r^\chi} \right) Y_l \right\|.$$

Hence, by Taylor's theorem

$$\left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| \leq \frac{1}{n_k^\chi} \left(\left\| \sum_{l < n_{k+1}} Y_l \right\| + \left\| \sum_{l < n_k} Y_l \right\| \right)$$

$$+ \sum_{r=n_k}^{n_{k+1}-2} \frac{r^\chi - (r+1)^\chi}{r^\chi (r+1)^\chi} \left\| \sum_{l < n_{k+1}} Y_l - \sum_{l \leq r} Y_l \right\|$$

$$\leq \frac{1}{n_k^\chi} \left(\left\| \sum_{l < n_{k+1}} Y_l \right\| + \left\| \sum_{l < n_k} Y_l \right\| \right)$$

$$(65) \qquad + \sum_{r \in I_k} \frac{\chi}{r^{\chi+1}} \left(\left\| \sum_{l < n_{k+1}} Y_l \right\| + \left\| \sum_{l \leq r} Y_l \right\| \right),$$

where the summations all start from $l = 1$ and stop at $l = n_k - 1$, r or $n_{k+1} - 1$. Furthermore, by the hypothesis and (33) we have that

$$
(66) \qquad \lim_{k \to \infty} \max_{r \in I_k} \frac{1}{r^\chi} \left\| \sum_{l < n_{k+1}} Y_l \right\| = 0
$$

and the first two terms on the RHS of (65) go to zero. Moreover, by (33)

$$
\sum_{r \in I_k} \frac{1}{r} \le \log \left(\frac{n_{k+1} - 1}{n_k - 1} \right)
$$

$$
(67) \qquad = \log \left(\frac{\lfloor (a(k+1))^{\frac{1}{1-\chi}} \rfloor - 1}{\lfloor (ak)^{\frac{1}{1-\chi}} \rfloor - 1} \right) \to 0 \text{ as } k \to \infty
$$

due to the fact that

$$
1 \le \frac{\lfloor (a(k+1))^{\frac{1}{1-\chi}} \rfloor - 1}{\lfloor (ak)^{\frac{1}{1-\chi}} \rfloor - 1} \le \frac{(a(k+1))^{\frac{1}{1-\chi}}}{(ak)^{\frac{1}{1-\chi}} - 2} = \frac{(\frac{k+1}{k})^{\frac{1}{1-\chi}}}{1 - (\frac{2}{ak})^{\frac{1}{1-\chi}}} \to 1 \text{ as } k \to \infty.
$$

In addition, by assumption, (66) and (67)

$$
\sum_{r \in I_k} \frac{\chi}{r^{\chi+1}} \left\| \sum_{l < n_{k+1}} Y_l \right\| \overset{k}{\ll} \sum_{r \in I_k} \frac{1}{r} \frac{1}{n_k^\chi} \left\| \sum_{l < n_{k+1}} Y_l \right\| \to 0 \text{ as } k \to \infty
$$

and

$$
\sum_{r \in I_k} \frac{\chi}{r^{\chi+1}} \left\| \sum_{l \le r} Y_l \right\| \le \sum_{r \in I_k} \frac{\chi}{r} \frac{1}{r^\chi} \left\| \sum_{l \le r} Y_l \right\| \to 0 \text{ as } k \to \infty
$$

Hence, the last term on the RHS of (65) goes to zero too.

iv) By Lemma 13, the fact that $\|B\| \le ||| B ||| \le \sqrt{d} \|B\|$ for a matrix with rank d, iii) and (38) we have

$$
\sum_{l \in I_k} \left\| \frac{\hat{A}_l}{l^\chi} \right\| \le \sum_{l \in I_k} \left\| \left| \frac{\hat{A}_l}{l^\chi} \right| \right\| \le \sqrt{d} \left\| \left| \sum_{l \in I_k} \frac{\hat{A}_l}{l^\chi} \right| \right\| \le d \left\| \sum_{l \in I_k} \frac{\hat{A}_l}{l^\chi} \right\|
$$

$$
\le d \left\| \sum_{l \in I_k} \frac{(\hat{A}_l - A)}{l^\chi} \right\| + d\|A\| \sum_{l \in I_k} \frac{1}{l^\chi}
$$

$$
(68) \qquad \le d \left\| \sum_{l \in I_k} \frac{Y_l}{l^\chi} \right\| + d\|A\| \sum_{l \in I_k} \frac{1}{l^\chi} \overset{k}{\ll} 1 \quad \text{for } k = 0, 1, 2, \dots
$$

Moreover, by (39) $\sum_{l \in I_k} \eta_l \le \bar{\eta} \sum_{l \in I_k} \frac{1}{l^\theta} \to 0$ as $k \to \infty$.

v) This follows by iv) and the fact that

$$
\prod_{l \in I_k} \left(1 + \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right) \le \exp \left(\sum_{l \in I_k} \frac{\|\hat{A}_l\|}{l^\chi} + \eta_l \right)
$$

$$
(69) \qquad \overset{k}{\ll} 1, \quad \forall k = 0, 1, \dots \ \square
$$

The following lemma is taken from Kouritzin [21].

LEMMA 13. *Suppose m is a positive integer and $\{M_k, k = 1, 2, 3, ...\}$ is a sequence of symmetric, positive semidefinite $R^{m \times m}$-matrices. Then, it follows that*

$$\sum_{k=1}^{j} ||| \, M_k \, ||| \leq \sqrt{m} \left|\left|\left| \sum_{k=1}^{j} M_k \right|\right|\right|, \qquad \forall j = 1, 2, 3, ...$$

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